Design of a Static Output Feedback H-Infinity Controller for Linear Time-Invariant Systems: an LMI Approach

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***Abstract*—This note is concerned with the static output feedback (SOF) H-infinity controller design for linear time-invariant systems. A sufficient bilinear matrix inequality (BMI) condition is developed for finding a stabilizing static output feedback H-infinity controller. For the first time, a novel cone complementary linearization approach within the context of linear matrix inequalities (LMI) is proposed to investigate the feasibility of the stabilizing controller synthesis along with a minimized H-infinity attenuation rate. The proposed method requires neither any apriori assumption generally made on the input/output matrices nor any application of a coordinate transformation. An example is presented for the application of the proposed scheme.**

Keywords—static output feedback, cone complementary, linear matrix inequalities

I. INTRODUCTION

The static output feedback control (SOF) approach receives quite many attraction, thus it is often addressed in control system design. In general, it may often not be very easy to access the inner structure of the plant under consideration in order to make an adjustment from the inside. Therefore, a feedback loop which takes the system output into a consideration is utilized to accommodate the performance of the system such that it fits to a desired output response. Output feedback is also quite reasonable to apply in practice under some circumstances where it is impossible to utilize state-feedback simply because the states can not be measured or at least some part of the state information is not accessible for some reasons. Though advantageous from this point of view, unlike state-feedback, output feedback control problem is a difficult issue since the synthesis conditions of this problem are non-convex and often appear in the form of bilinear matrix inequality (BMI) which makes it hard to find feasible solution set, say, via a linear matrix inequality (LMI) solver.

The H_∞ control problem was studied in [1] for linear uncertain systems. Unlike a Riccati inequality based approach, the negativeness of a matrix was taken into consideration in [1]. Concerning the linear uncertain continuous-time systems, an H_∞ controller was developed in [3] based on a two-stage LMI approach. In the first stage, a parameter dependent state-feedback controller was obtained to be utilized for synthesizing the robust output feedback H_∞ dynamic controller [3]. Some sufficient dilated LMI conditions have been derived in [4] for the robust H_∞ SOF stabilization of linear uncertain continuous-time systems. The synthesis of a convex SOF controller for linear uncertain systems was introduced in [5] and the proposed method has also been extended to the case of H_∞ control design. In [6], the H_∞ control problem was investigated for linear continuous-time systems by employing a two-step LMI based procedure for an achievement of a desired disturbance attenuation performance and the sensitivity performance simultaneously. The SOF H_∞ controller design has been investigated in [7] for linear uncertain systems. The developed scheme has given LMI conditions which led to less conservative results [7]. Based on a new structure for the auxiliary matrix variables yielding a relaxation on the LMI conditions, the robust reduced-order controller design has been presented in [8]. In order to penalise the number of nonzero entries of the SOF gain, an extra term has been incorporated in [9] into the optimisation objective function and the structured H_2 and H_∞ SOF problem has been solved accordingly. Finally, utilizing a descriptor form of system representation which allows to introduce free slack matrices in the design condition, a robust SOF H_∞ control has been developed for linear uncertain continuous-time systems [10].

This note takes into account the SOF H_∞ control problem for linear time-invariant continuous-time systems. The proposed method relies on transforming the BMI condition into a new BMI form with the use of completing the cross terms to squared quadratic terms. This novel approach allows to formulate the achievement of a prospective feasible solution set through some well-known linearization techniques. To the best of author's knowledge, for the very first time, the so-called cone complementary nonlinear minimization method is thus applied for designing an SOF H_∞ controller. An example is demonstrated to show the application of the proposed scheme.

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II. Problem Statement

We take into account a class of linear time-invariant continuous-time control system defined in (1):

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_w w(t) + B_u u(t) \\ z(t) &= C_z x(t) + D_w w(t) + D_u u(t) \\ y(t) &= C_y x(t) + E_w w(t) \end{aligned} \tag{1}$$

where $x(t) \in \Re^n$, $w(t) \in \Re^q$ is an exogeneous noise signal in $L_2 \in [0, \infty)$, $u(t) \in \Re^m$ is the control signal, $z(t) \in \Re^r$ is the regulated output, $y(t) \in \Re^p$ is the measured output. The matrices A, B_w, B_u, C_z, D_w are system matrices which are all konown. A SOF control law is chosen for system (1) as follows:

$$u(t) = Fy(t) \tag{2}$$

where $F \in \Re^{m \times p}$ is a suitably chosen SOF gain matrix. The dynamics of the system can be obtained by substituting (2) into (1) as

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_{wc} w(t) \\ z(t) &= C_{zc} x(t) + D_{wc} w(t) \\ y(t) &= C_y x(t) + E_w w(t) \end{aligned} \tag{3}$$

where $A_c = A + B_u F C_y$, $B_{wc} = B_w + B_u F E_w$, $C_{zc} = C_z + D_u F C_y$, $D_{wc} = D_w + D_u F E_w$. The objective of the SOF H_∞ control problem can be stated as achieving the goal of finding an H_∞ controller of the form given in (2) such that the two conditions described in [11] are fulfilled accordingly.

III. Main Results

A novel matrix inequality condition is presented for finding an SOF H_∞ controller summarized.

Theorem 1. Given a positive scalar $\gamma > 0$, if the matrices $0 < P^T = P \in \Re^{n \times n}$, $0 < Z^T = Z \in \Re^{m \times m}$, and a matrix $F \in \Re^{m \times p}$ exist such that

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} & \Sigma_{16} & 0 \\ * & -\gamma^2 \mathrm{I}_q & 0 & 0 & \Sigma_{25} & 0 & \Sigma_{27} \\ * & * & -\mathrm{I}_n & 0 & 0 & 0 & 0 \\ * & * & * & -\mathrm{I}_m & 0 & 0 & 0 \\ * & * & * & * & -\mathrm{I}_r & 0 & 0 \\ * & * & * & * & * & -Z & 0 \\ * & * & * & * & * & * & -Z^{-1} \end{bmatrix} < 0 \tag{4}$$

where $\Sigma_{11} = -A^T A - P\left(\mathrm{I}_n + B_u B_u^T\right) P$, $\Sigma_{12} = PB_w$, $\Sigma_{13} = A^T + P$, $\Sigma_{14} = PB_u + C_y^T F^T$, $\Sigma_{15} = C_z^T + C_y^T F^T D_u^T$, $\Sigma_{16} = PB_u$, $\Sigma_{25} = D_w^T + E_w^T F^T D_u^T$, $\Sigma_{27} = E_w^T F^T$ and (*) denotes the terms due to symmetry and I_n is an identity matrix of n by n, then the controller in (2) with F becomes an SOF H_∞ controller for system (1), (2).

Proof. We shall choose a candidate Lyapunov function as follows

$$V(x(t)) = x^T(t) P x(t) \tag{5}$$

We compute $\dot{V}(x(t))$ as

$$\begin{aligned} \dot{V}(x(t)) &= 2x^T(t) P \dot{x}(t) \\ &= 2x^T(t) P \left[A_c x(t) + B_{wc} w(t) \right] \\ &= x^T(t) \left(PA_c + A_c^T P \right) x(t) + 2x^T(t) P B_{wc} w(t) \end{aligned} \tag{6}$$

then we calculate the following quadratic expression in (7)

$$\begin{aligned} &\dot{V}(x(t)) + z^T(t) z(t) - \gamma^2 w(t) w(t) \\ &\quad = x^T(t) \left(PA_c + A_c^T P + C_{zc}^T C_{zc} \right) x(t) \\ &\qquad + 2x^T(t) \left(PB_{wc} + C_{zc}^T D_{wc} \right) w(t) \\ &\qquad + w^T(t) \left(-\gamma^2 \mathrm{I}_q + D_{wc}^T D_{wc} \right) w(t) \\ &\quad = \chi^T(t) \Omega_0 \chi(t) \end{aligned} \tag{7}$$

where $\chi(t) = \begin{bmatrix} x^T(t) & w^T(t) \end{bmatrix}^T$ and $\Omega_0 = \begin{bmatrix} PA_c + A_c^T P + C_{zc}^T C_{zc} & PB_{wc} + C_{zc}^T D_{wc} \\ * & -\gamma^2 \mathrm{I}_q + D_{wc}^T D_{wc} \end{bmatrix}$.

If the following matrix inequality

$$\Omega_0 < 0 \tag{8}$$

holds true, then we obtain

$$\dot{V}(x(t)) + z^T(t) z(t) - \gamma^2 w(t) w(t) = \chi^T(t) \Omega_0 \chi(t) < 0 \tag{9}$$

We now consider $PA_c + A_c^T P$ in Ω_0 as follows

$$\begin{aligned} PA_c + A_c^T P &= PA + A^T P + PB_u F C_y + \left(PB_u F C_y \right)^T \\ &= -A^T A - PP + \left(A + P \right)^T \left(A + P \right) \\ &\quad - PB_u B_u^T P + \left(PB_u + C_y^T F^T \right)^T \left(PB_u + C_y^T F^T \right) \\ &\quad - C_y^T F^T C_y F \end{aligned} \tag{10}$$

We shall now substitute (10) into (8) and apply Schur complement to obtain

$$\Omega_1 < 0 \tag{11}$$

where $\Omega_1 = \begin{bmatrix} \Sigma_{11} & \Omega_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} \\ * & -\gamma^2 \mathrm{I}_q & 0 & 0 & \Sigma_{25} \\ * & * & -\mathrm{I}_n & 0 & 0 \\ * & * & * & -\mathrm{I}_m & 0 \\ * & * & * & * & -\mathrm{I}_r \end{bmatrix}$ with $\Omega_{12} = PB_w + PB_u F E_w$. In order to linearize the matrix inequality in (11) with respect to the SOF gain matrix F we shall reexpress (11) via decomposition as follows

$$\Omega_1 = \Psi + \Pi^T\Theta + \Theta^T\Pi < 0 \tag{12}$$

where $\Psi = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} \\ * & -\gamma^2 \mathrm{I}_q & 0 & 0 & \Sigma_{25} \\ * & * & -\mathrm{I}_n & 0 & 0 \\ * & * & * & -\mathrm{I}_m & 0 \\ * & * & * & * & -\mathrm{I}_r \end{bmatrix}$ and

$\Pi = \begin{bmatrix} B_u^T P & 0 & 0 & 0 & 0 \end{bmatrix}^T$, $\Theta = \begin{bmatrix} 0 & FE_w & 0 & 0 & 0 \end{bmatrix}$.

Employing the well-known bounding inequality in (12) yields

$$\Psi + \Pi^T\Theta + \Theta^T\Pi < \Psi + \Pi^T Z^{-1}\Pi + \Theta^T\left(Z^{-1}\right)^{-1}\Theta < 0 \tag{13}$$

Applying Schur complement to (13) allows to obtain the BMI given in (4). If $w(t) = 0$, is guaranteed if (8) is ensured which indicates that system (3) is asymptotically stable implying that the aforementioned first condition of the objective of the present work is satisfied. Integrating both sides of (9) from 0 to ∞ leads to obtain

$$\begin{aligned} &\int_0^\infty \dot{V}(x(t))\,dt = \lim_{t\to\infty} V(x(t)) - V(x(0)) \\ &< \int_0^\infty \left[-z^T(t)z(t) + \gamma^2 w^T(t)w(t)\right]dt \end{aligned} \tag{14}$$

Assuming that $x(0) = 0$, thus yielding $V(x(0)) = 0$ and as $\lim_{t\to\infty} V(x(t)) > 0$, we obtain

$$\int_0^\infty z^T(t)z(t)\,dt = \|z(t)\|_2^2 < \gamma^2 \|w(t)\|_2^2 = \gamma^2 \int_0^\infty w^T(t)w(t)\,dt \tag{15}$$

As a result, the condition (2) [11] of the objective of the present work is also achieved as shown in (15). This completes the proof.

If we assume that the SOF gain matrix is fixed, then the following Corollary is presented for finding a minimum allowable attenuation rate of γ.

Corollary 1. Given a positive scalar $\gamma > 0$ and a matrix $F \in \Re^{m\times p}$, if the matrix $0 < P^T = P \in \Re^{n\times n}$ satisfying the linear matrix inequality of (8) then the controller in (2) with F becomes an SOF H_∞ controller for system (1), (2).

Proof. If F is fixed, then the matrix inequality in (8) becomes an LMI which then follows with the proof of Theorem 1 from (5) to (9).

Note that the synthesis condition for a stabilizing SOF H_∞ controller given in (4) is not convex, we shall consider the use of an iterative algorithm to resolve the feasibility problem of (4). We can find a matrix $0 < L^T = L \in \Re^{n\times n}$ such that

$$-P\left(\mathrm{I}_n + B_u B_u^T\right)P \le -L \tag{16}$$

The inequality in (16) is equivalent to

$$L^{-1} \ge P^{-1}\left(\mathrm{I}_n + B_u B_u^T\right)P^{-1} \tag{17}$$

Applying Schur complement to (17) yields

$$\begin{bmatrix} L^{-1} & P^{-1} \\ * & \mathrm{I}_n + B_u B_u^T \end{bmatrix} \ge 0 \tag{18}$$

Introducing new variables M, X such that $M = L^{-1}$, $X = P^{-1}$ allows to rewrite (18)

$$\begin{bmatrix} M & X \\ * & \mathrm{I}_n + B_u B_u^T \end{bmatrix} \ge 0 \tag{19}$$

In a similar manner, there exists a real and symmetric and positive definite matrix $0 < Q^T = Q \in \Re^{m\times m}$ such that $-Z^{-1} \le -Q$ which is equivalent to $Q^{-1} - Z \ge 0$ and by Schur complement, the former inequality is satisfied equivalently by

$$\begin{bmatrix} N & Z \\ * & Z \end{bmatrix} \ge 0 \tag{20}$$

where N is introduced such that $N = Q^{-1}$. Now we employ the well-known cone complementary technique [12] which leads us to present the following nonlinear minimization algorithm by replacing the original nonconvex feasibility condition with some LMI conditions as follows:

$$\text{Minimize } \operatorname{trace}(LM + PX) + \operatorname{trace}(QN)$$

$$\text{subject to (19), (20), } \begin{bmatrix} L & \mathrm{I}_n \\ * & M \end{bmatrix} \ge 0, \begin{bmatrix} P & \mathrm{I}_n \\ * & X \end{bmatrix} \ge 0,$$

$$\begin{bmatrix} Q & \mathrm{I}_m \\ * & N \end{bmatrix} \ge 0, \text{ and}$$

$$\left.\begin{bmatrix} \Psi & PB_u & 0 \\ * & -Z & E_w^T F^T \\ * & * & -Q \end{bmatrix}\right|_{-P\left(\mathrm{I}_n + B_u B_u^T\right)P \to -L} < 0 \tag{21}$$

As a result, once we obtain a solution of $2n + m$ for the aforementioned minimization problem, we shall conclude that the linear time-invariant system with (2) is asymptotically stable within the context of H_∞ control via achieving a minimum attenuation rate of γ by employing a similar form of algorithm whose detals are outlined in [11].

IV. Numerical Results

We now present an example from the literature for the application of the proposed stabilizing SOF H_∞ controller along with a minimum achieved attenuation rate of γ.

Example 1: We shall let the parameters of system (1) described as

$$A=\begin{pmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.7070 & 1.4200 \\ 0 & 0 & 1.0000 & 0 \end{pmatrix},$$

$$B_w=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ B_u=\begin{pmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{pmatrix},$$

$$C_z=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ D_w=\begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$D_u=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ C_z=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$E_w=\begin{pmatrix} 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$$

Utilizing Theorem 1 and Corollary 1, we have resolved the feasibility problem for a suboptimal minimum allowable attenuation rate of γ shown in Table I. In order to make a comparison with a full-order controller that is when $u(t)=Kx(t)$ where $K\in\Re^{m\times n}$ is utilized, the H_∞ performance is also presented in Table I. It can be seen that the H_∞ performance of the proposed reduced-order state-feedback that is SOF controller is quite close to that of a full-order state-feedback controller. In other words, though using few number of state variables in the construction of the feedback controller and thus reducing the cost of the control input, we notice that the proposed controller is still capable of exhibiting almost equivalent level of H_∞ performance. Moreover, the entries of the static output feedback gain matrix, F is significantly smaller than those of the state-feedback gain matrix, K.

TABLE I. MINIMUM ALLOWABLE ATTENUATION RATE

Methods	γ, F/K
Theorem 1	1.5097, $F=\begin{pmatrix} -0.1144 & -1.3906 \\ -0.0986 & 2.3959 \end{pmatrix}$
Corollary 1 with $F=\begin{pmatrix} -0.1144 & -1.3906 \\ -0.0986 & 2.3959 \end{pmatrix}$	1.1741
Full-order state feedback controller	1.1104, $K=\begin{pmatrix} -697.056 & 48.6121 \\ -55.5743 & 4.0701 \\ 52.7125 & 383.88 \\ 4.3048 & 29.8743 \end{pmatrix}$

V. CONCLUSIONS

This note has investigated an H_∞ reduced-order SOF controller on the basis of Lyapunov stability theory. The synthesis of a stabilizing H_∞ SOF controller has been accomplished by employing the cone complementary approach for the very first time. Utilizing a linearization approach along with an iterative cone complementary technique has allowed to introduce a set of LMI conditions replacing the original BMI condition. An example has been taken into account for the application of the developed method. The numerical results have clearly shown that the proposed stabilization method has yielded an H_∞ performance remaining comparably quite close to that achieved by the use of a full-order state-feedback controller.

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